Exercise 63

Prove that f is continuous at a if and only if

$$\lim_{h \to 0} f(a+h) = f(a)$$

Solution

Suppose that f is continuous at a. Then by the definition of continuity,

$$\lim_{x \to a} f(x) = f(a).$$

Make the substitution, $h = x - a$, to get the desired result. Note that as $x \to a$, $h \to 0$.

$$\lim_{h \to 0} f(h+a) = f(a)$$

Suppose instead that

$$\lim_{h \to 0} f(h+a) = f(a).$$

Make the substitution, $x = h + a$. Note that as $h \to 0$, $x \to a$.

$$\lim_{x \to a} f(x) = f(a)$$

This equation indicates that $f(x)$ is continuous at a. Therefore, f is continuous at a if and only if

$$\lim_{h \to 0} f(a+h) = f(a).$$